On the linearity of the periods of subtraction games

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Abstract

A subtraction game is an impartial combinatorial game involving a finite set S of positive integers. The nim-sequence $\mathcal{G}_S$ associated with this game is ultimately periodic. In this paper, we study the nim-sequence $\mathcal{G}_{S\cup\{c\}}$ where S is fixed and c varies. We conjecture that there is a multiple q of the period of $\mathcal{G}_S$, such that for sufficiently large c, the pre-period and period of $\mathcal{G}_{S\cup\{c\}}$ are linear in c if c modulo q is fixed. We prove it in several cases.

We also give new examples with period 2 inspired by this conjecture.

1. Introduction

Let S be a finite set of positive integers. The *(finite) subtraction game* $\mathrm{SUB}(S)$ is a two-player game involving a heap of $n \geq 0$ counters. The two players move alternately, subtracting some $s \in S$ counters. The player who cannot make a move loses.

We always write the subtraction set as $S = \{s_1, \dots, s_k\}$ with an order $s_1 < s_2 < \cdots < s_k$. Denote by $\mathcal{G}(n) = \mathcal{G}_S(n)$ the *nim-value* (or *Grundy-value*), i.e.,

$$\mathcal{G}(n) = \operatorname{mex}\{\mathcal{G}(n-s) : s \in S, s \leq n\}, \quad \forall n \geq 0,$$

where mex means the minimal non-negative integer not in the set. The sequence $\mathcal{G} = \mathcal{G}_S = \{\mathcal{G}(n)\}_{n\geq 0}$ is called the *nim-sequence* (or *Sprague-Grundy sequence*).

If $d = \gcd(S) = \gcd\{s : s \in S\} > 1$ and $S' = \{s/d : s \in S\}$, then $\mathcal{G}_S(n) = \mathcal{G}_{S'}(m)$, where $md \leq n < (m+1)d$. Hence we may assume that $\gcd(S) = 1$ if necessary.

Definition 1. A subtraction game $\mathrm{SUB}(S)$ (or its nim-sequence $\mathcal{G}$) is called *ultimately periodic*, if there exist integers $p \geq 1$ and $\ell \geq 0$ such that $\mathcal{G}(n+p) = \mathcal{G}(n)$ for all $n \geq \ell$. The minimal p is called the *period* and the minimal ℓ is called the *pre-period*.

Since $\mathcal{G}(n) \leq k$, one can show that $\mathcal{G}$ is ultimately periodic with $\ell, p \leq (k+1)^{s_k}$ by the pigeonhole principle, see [1, Theorem 7.33].

Since $\mathcal{G}(n + s_k)$ only depends on $\mathcal{G}(n), \mathcal{G}(n+1), \dots, \mathcal{G}(n+s_k-1)$, we have the following lemma to determine the period and pre-period.

Lemma 1.1 ([1, Corollary 7.34]). *The minimal integers $\ell \geq 0, p \geq 1$ such that $\mathcal{G}(n) = \mathcal{G}(n+p)$ for $\ell \leq n < \ell + s_k$ are the pre-period and period of $\mathrm{SUB}(S)$ respectively.*

The nim-sequence $\mathcal{G}$ is known when $k \leq 2$. For $k \geq 3$, even the pre-period and the period are not known in general. In §§2-3, we will recall some known results with $k \leq 3$, and give several new results with $k = 3$. We also give the nim sequence when $k \geq 4$ and S have a special form in §4. Based on these results and some computer-assistant calculations, we propose a conjecture on the inductive behavior of ℓ and p as follows:

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Conjecture 1.2 (Asymptotic linearity). *Fix a subtraction set S. Then the pre-period and the period of the* $\textsc{sub}(S \cup \{c\})$ *grow at most linearly in c.*

Moreover, the pre-period and the period should increase piecewise linearly on c:

Conjecture 1.3 (Piecewise linearity). *Fix a subtraction set S. There are*

- *positive integers q, N;*
- *integers $\alpha_r, \beta_r, \lambda_r, \mu_r$ for each $0 \le r < q$,*

such that if $c \geq N$ and $c \equiv r \bmod q$,

- *the pre-period of* $\textsc{sub}(S \cup \{c\})$ *is $\alpha_r c + \beta_r$;*
- *the period of* $\textsc{sub}(S \cup \{c\})$ *is $\lambda_r c + \mu_r$.*

In many cases, q is the period of $\textsc{sub}(S)$.

Theorem 1.4. *Conjecture 1.3 holds in the following cases:*

1. *$1 \in S$ and the elements of S are all odd;*
2. *$S = \{1, b\}$;*
3. *$S = \{a, 2a\}$;*
4. *$S = \{a, a+1, \ldots, b-1, b\}$.*

We will also give new ultimately bipartite nim-sequences inspired by this conjecture. See Theorem 6.3.

Remark 1. Once Conjecture 1.3 holds with effective q, N, then one can get the pre-period and period of $\textsc{sub}(S \cup \{c\})$ for all c effectively. That is because we only need to calculate the pre-periods and periods of $\textsc{sub}(S \cup \{c\})$ for $c \le N + 2q$.

Remark 2. Denote by $\mathcal{P}(n) \in \{0, 1\}$ the sign of $\mathcal{G}(n)$. Then $\mathcal{P}(n) = 1$ if and only if the starting position with heap size n is a win for the player to move. One can easily see that $\mathcal{P}$ is ultimately periodic with pre-period $\le \ell$, period $\le p$ and both of them $\le 2^{s_k}$. We can propose a similar conjecture on the $\mathcal{P}$-sequence of $\textsc{sub}(S \cup \{c\})$, which is a consequence of Conjecture 1.3.

Remark 3. In [2], Althöfer and Bültermann studied the pre-period and period of the $\mathcal{P}$-sequence of $\textsc{sub}(S)$, where all elements of S are linear in a variable s. For example, they conjectured that $\textsc{sub}(s, 4s, 12s+1, 16s+1)$ has no pre-period and period $56s^3 + 52s^2 + 9s + 1$. Our conjecture is in a different direction since we do not require the subtraction set $S \cup \{c\}$ to have a special form.

Let's introduce some notations we will use. Let t, a be non-negative integers and $\mathcal{H} = (h_1 \cdots h_k)$ a sequence of integers with finite length. As usual, we denote by a^t the sequence $a \cdots a$ (t copies of a) and $\mathcal{H}^t$ the sequence $\mathcal{H} \cdots \mathcal{H}$ (t copies of $\mathcal{H}$). Denote by $\underline{\mathcal{H}}$ the infinite-length sequence with periodic sequence $\mathcal{H}$, i.e., $\underline{\mathcal{H}} = \mathcal{H}\mathcal{H} \cdots$. For example, if a nim-sequence $\mathcal{G}$ has pre-period ℓ and period p, then we can write

$$\mathcal{G} = \mathcal{G}(0)\mathcal{G}(1)\mathcal{G}(2) \cdots = \mathcal{G}(0) \cdots \mathcal{G}(\ell - 1)\underline{\mathcal{G}(\ell) \cdots \mathcal{G}(\ell + p - 1)}.$$

We will not give detailed proofs of all nim-sequences, since these proofs tend to involve lengthy and tedious inductions.

2. The case $S = \{1, b, c\}$

In this section, we will consider the nim-sequences of $S = \{1, b, c\}$, where $1 < b < c$. Let's recall some classical cases first.

Lemma 2.1. *Let p be the period of* SUB(S)*. Let $S' = S \cup \{x + pt\}$ for some $x \in S$ and $t \geq 1$. If the pre-period of* SUB(S) *is zero, then $\mathcal{G}_{S'} = \mathcal{G}_S$.*

PROOF. Certainly $\mathcal{G}_{S'}(0) = \mathcal{G}_S(0) = 0$. Suppose that $\mathcal{G}_{S'}(i) = \mathcal{G}_S(i)$ for $0 \leq i \leq n-1$. If $n < x + pt$, then

$$\mathcal{G}_{S'}(n) = \text{mex}\,\{\mathcal{G}_S(n-s) : s \in S, s \leq n\} = \mathcal{G}_S(n).$$

If $n \geq x + pt$, then

$$\begin{aligned}\mathcal{G}_{S'}(n) &= \text{mex}\,\{\mathcal{G}_S(n-x-pt), \mathcal{G}_S(n-s) : s \in S, s \leq n\} \\ &= \text{mex}\,\{\mathcal{G}_S(n-x), \mathcal{G}_S(n-s) : s \in S, s \leq n\} = \mathcal{G}_S(n).\end{aligned}$$

The lemma then follows by induction.

Example 2.2. *Certainly, $\mathcal{G}_{\{1\}} = \underline{01}$. If $1 \in S$ and all elements of S are odd, then $\mathcal{G}_S = \underline{01}$ by applying Lemma 2.1 several times. This condition is also necessary for $\mathcal{G}_S = \underline{01}$, see [4].*

Example 2.3. *Let $S = \{a, c\}$ with $1 \leq a < c$. Write $c = at + r, 0 \leq r < a$. Then*

$$\mathcal{G}_S = \begin{cases} \underline{(0^a 1^a)^{t/2} 0^r 2^{a-r} 1^r}, & \text{if } t \text{ is even;} \\ \underline{(0^a 1^a)^{(t+1)/2} 2^r}, & \text{if } t \text{ is odd,} \end{cases}$$

$\ell = 0$ and $p = c + a$ or $2a$. See [3] and [2, Theorem 2].

Example 2.4. 1. *Let $S = \{1, b, c\}$ with odd b and $1 < b < c$. Note that $\mathcal{G}_{\{1,b\}} = \underline{\mathcal{H}}$ where $\mathcal{H} = 01$. We have*

c	$\mathcal{G}_S$	ℓ	p
odd	$\underline{\mathcal{H}}$	0	2
even	$\underline{\mathcal{H}^{c/2}(23)^{(b-1)/2}2}$	0	$c+b$

See [5, Theorem 4].

2. *Let $S = \{1, 2, c\}$ with $c > 2$. Note that $\mathcal{G}_{\{1,2\}} = \underline{\mathcal{H}}$ where $\mathcal{H} = 012$. Write $c = 3t + r, 0 \leq r < 3$. Then*

r	$\mathcal{G}_S$	ℓ	p
0	$\underline{(012)^t 3}$	0	$c+1$
1, 2	$\underline{012}$	0	3

3. *Let $S = \{1, 4, c\}$ with $c > 4$. Note that $\mathcal{G}_{\{1,4\}} = \underline{\mathcal{H}}$ where $\mathcal{H} = 01012$. Write $c = 5t + r, 0 \leq r < 5$. Then*

r, c	$\mathcal{G}_S$	ℓ	p
$r = 0, c = 5$	$\underline{\mathcal{H}\,323}$	0	8
$r = 0, c > 5$	$\mathcal{H}^t\,323013\underline{\mathcal{H}^{t-1}012012}$	$c+6$	$c+1$
$r = 1, 4$	$\underline{\mathcal{H}}$	0	5
$r = 2$	$\underline{\mathcal{H}^t\,012}$	0	$c+1$
$r = 3$	$\underline{\mathcal{H}^{t+1}\,32}$	0	$c+4$

Proposition 2.5. *Let $S = \{1, b, c\}$ with even $b = 2k \geq 6$. Write $c = t(b+1) + r$ with $0 \leq r \leq b, t \geq 1$.*

1. *If $r = 1, b$, then $\ell = 0$ and $p = b + 1$.*
2. *If $3 \leq r \leq b - 1$ is odd, then $\ell = 0$ and $p = c + b$.*
3. *If $r = b - 2$, then $\ell = 0$ and $p = c + 1$.*
4. *If $c = b + 1$, then $\ell = 0, p = 2b$;*
5. *If $c > b + 1$, $0 \leq r \leq b - 4$ is even and $t + r/2 \geq k$, then $\ell = \left(\frac{b-r}{2} - 1\right)(c + b + 2) - b$ and $p = c + 1$.*
6. *If $c > b + 1$, $0 \leq r \leq b - 4$ is even and $t + r/2 \leq k - 1$, then $\ell = t(c + b + 2) - b$. If $t + r/2 < k - 1$, then $p = c + b$; if $t + r/2 = k - 1$, then $p = b - 1$.*

Proof. Note that $\mathcal{G}_{\{1,b\}} = \underline{\mathcal{H}}$ where $\mathcal{H} = (01)^k 2$.

1. In this case, $\mathcal{G} = \underline{\mathcal{H}}$, $\ell = 0$ and $p = b + 1$ by Lemma 2.1.

2. In this case, $\mathcal{G} = \underline{\mathcal{H}^{t+1}\,(32)^{(r-1)/2}}$, $\ell = 0$ and $p = c + b$.

3. In this case, $\mathcal{G} = \underline{\mathcal{H}^t\,(01)^{k-1}2}$, $\ell = 0$ and $p = c + 1$.

4. In this case, $\mathcal{G} = \underline{(01)^k(23)^k} = \mathcal{H}3(23)^{k-1}$, $\ell = 0$ and $p = 2b$.

5. Write $r = 2v$. If $1 \leq v \leq k - 2$, the leading $(c + 1)(k - v + 1)$ terms of $\mathcal{G}$ are (the waved part is the first periodic nim-sequence)

i	$\mathcal{G}\big((c+1)i + j\big),\ 0 \leq j \leq c$
0	$\mathcal{H}^t, (01)^v 2$
1	$(32)^{k-v-1}(01)^{v+1}2, \mathcal{H}^{t-1}, (01)^v 0$
2	$1(01)^{k-v-2}2(01)^{v+1}2, (32)^{k-v-2}(01)^{v+2}2, \mathcal{H}^{t-2}, (01)^v 0$
i	$1(01)^{k-v-2}2(01)^{v+1}0, \ldots, 1(01)^{k-v-i+1}2(01)^{v+i-2}0,$ $1(01)^{k-v-i}2(01)^{v+i-1}2, (32)^{k-v-i}(01)^{v+i}2, \mathcal{H}^{t-i}, (01)^v 0$
$k - v - 1$	$1(01)^{k-v-2}2(01)^{v+1}0, \ldots, 1(01)^2 2(01)^{k-3}0, 1(01)2(01)^{k-2}2,$ $(32)\underline{(01)^{k-1}2, \mathcal{H}^{t-k+v+1}, (01)^v 0}$
$k - v$	$\underline{1(01)^{k-v-2}2(01)^{v+1}0, \ldots, 1(01)2(01)^{k-2}0,}\ 12(01)^{k-1}2,$ $\mathcal{H}^{t-k+v-1}, (01)^v 0.$

If $v = 0$, the leading $(c + 1)(k + 1)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((c+1)i + j\big),\ 0 \leq j \leq c$
0	$\mathcal{H}^t 3$
1	$(23)^{k-1}013, \mathcal{H}^{t-1}0$
2	$1(01)^{k-2}2(01)2, (32)^{k-2}(01)^2 2, \mathcal{H}^{t-2}0$
i	$1(01)^{k-2}2(01)0, \cdots, 1(01)^{k-i+1}2(01)^{i-2}0, 1(01)^{k-i}2(01)^{i-1}2,$ $(32)^{k-i}(01)^i 2, \mathcal{H}^{t-i}0$
$k - 1$	$1(01)^{k-2}2(01)0, \cdots, 1(01)^2 2(01)^{k-3}0, 1(01)^1 2(01)^{k-2}2,$ $(32)\underline{(01)^{k-1}2, \mathcal{H}^{t-k+1}0}$
k	$\underline{1(01)^{k-2}2(01)0, \cdots, 1(01)^1 2(01)^{k-2}0,}\ 12(01)^{k-1}2, \mathcal{H}^{t-k+1}0.$

In both cases, we have $\ell = \left(\frac{b-r}{2} - 1\right)(c + b + 2) - b$, $p = c + 1$ and

$$\mathcal{G} = \cdots \underline{2(01)^{k-1}\big(2(01)^k\big)^{t-k+v+1}\big(2(01)^{k-1}\big)^{k-v-1}}.$$

6. If $1 \leq v \leq k - 2$, the leading $(c + 1)(t + 2)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((c+1)i+j\big), 0 \le j \le c$
0	$\mathcal{H}^t\,(01)^v 2$
1	$(32)^{k-v-1}(01)^{v+1}2, \mathcal{H}^{t-1}(01)^v 0$
2	$1(01)^{k-v-2}2(01)^{v+1}2, (32)^{k-v-2}(01)^{v+2}2, \mathcal{H}^{t-2}(01)^v 0$
i	$1(01)^{k-v-2}2(01)^{v+1}0, \dots, 1(01)^{k-v-i+1}2(01)^{v+i-2}0,$ $1(01)^{k-v-i}2(01)^{v+i-1}2, (32)^{k-v-i}(01)^{v+i}2, \mathcal{H}^{t-i}(01)^v 0$
$t-1$	$1(01)^{k-v-2}2(01)^{v+1}0, \dots, 1(01)^{k-v-t+2}2(01)^{v+t-3}0,$ $1(01)^{k-v-t+1}2(01)^{v+t-2}2, (32)^{k-v-t+1}(01)^{v+t-1}2, \mathcal{H}^1(01)^v 0$
t	$1(01)^{k-v-2}2(01)^{v+1}0, \dots, 1(01)^{k-v-t+1}2(01)^{v+t-2}0,$ $1(01)^{k-v-t}2(01)^{v+t-1}2, (32)^{k-v-t}(01)^{v+t}2, (01)^v 0$
$t+1$	$1(01)^{k-v-2}2(01)^{v+1}0, \dots, 1(01)^{k-v-t+1}2(01)^{v+t-2}0,$ $1(01)^{k-v-t}2(01)^{v+t-1}0, 1(01)^{k-v-t-1}2(01)^{v+t}2, (32)^{k-v-t-1}01 \cdots$

Therefore, $\ell = t(c+b+2)-b$. If $t+v < k-1$, then $p = c+b$ and

$$\mathcal{G} = \cdots \underline{2(32)^{k-v-t-1}(01)^{v+t}2\big((01)^{k-1}2\big)^t(01)^{v+t}}.$$

If $t+v = k-1$, then $p = b-1$ and $\mathcal{G} = \cdots \underline{2(01)^{k-1}}$.

If $v = 0$, the leading $(c+1)(t+2)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((c+1)i+j\big), 0 \le j \le c$
0	$\mathcal{H}^t\,3$
1	$(23)^{k-1}013, \mathcal{H}^{t-1}0$
2	$1(01)^{k-2}2(01)2, (32)^{k-2}(01)^2 2, \mathcal{H}^{t-2}0$
i	$1(01)^{k-2}2(01)0, \cdots, 1(01)^{k-i+1}2(01)^{i-2}0, 1(01)^{k-i}2(01)^{i-1}2,$ $(32)^{k-i}(01)^i 2, \mathcal{H}^{t-i}0$
$t-1$	$1(01)^{k-2}2(01)0, \cdots, 1(01)^{k-t+2}2(01)^{t-3}0, 1(01)^{k-t+1}2(01)^{t-2}2,$ $(32)^{k-t+1}(01)^{t-1}2, \mathcal{H}^1 0$
t	$1(01)^{k-2}2(01)0, \cdots, 1(01)^{k-t+1}2(01)^{t-2}0, 1(01)^{k-t}2(01)^{t-1}2,$ $(32)^{k-t}(01)^t 20$
$t+1$	$1(01)^{k-2}2(01)0, \cdots, 1(01)^{k-t}2(01)^{t-1}0, 1(01)^{k-t-1}2(01)^t 0,$ $1(01)^{k-t-1}2(01)^t 2, (32)^{k-t-1}01 \cdots$

Therefore, $\ell = t(c+b+2)-b$. If $t < k-1$, then $p = c+b$ and

$$\mathcal{G} = \cdots \underline{2(32)^{k-t-1}(01)^t 2\big((01)^{k-1}2\big)^t(01)^t}.$$

If $t = k-1$, then $p = b-1$ and $\mathcal{G} = \cdots \underline{2(01)^{k-1}}$.

Remark 4. The case $c < 4b$ is studied in [5], but there are some incorrect data. In Table 1, $p = a-1$ if $r = a-3 \ge 3$. In Table B.11, $n_0 = a+2b+4$ if $2 \le r \le a-4$. In Table B.12, $n_0 = 2a+3b+6$ if $3 \le r \le a-5$. The corresponding pre-period nim-values also need to be modified.

3. The case $S = \{a, 2a, c\}$

Proposition 3.1. *Let $S = \{a, 2a, c\}$ with $2a < c$. Write $c = 3at + r$ with $0 \le r < 3a$. Then*

$$\ell = \begin{cases} c + a - r, & 0 < r < a; \\ 0, & \textit{otherwise}, \end{cases} \qquad p = \begin{cases} 3a/2, & r = a/2; \\ 3a, & a/2 < r \le 2a; \\ c + a, & \textit{otherwise}. \end{cases}$$

PROOF. Denote by $\mathcal{H} = 0^a 1^a 2^a$. Then $\mathcal{G}_{\{a,2a\}} = \underline{\mathcal{H}}$ with period $q = 3a$. Write $a = 2k - 1$ if a is odd; $a = 2k$ if a is even.

1. If $a \le r \le 2a$, then $\mathcal{G} = \underline{\mathcal{H}}$, $\ell = 0$ and $p = 3a$.
2. If $r = 0$, then $\mathcal{G} = \underline{\mathcal{H}^t 3^a}$, $\ell = 0$ and $p = c + a$.
3. If $0 < r < k$, then

$$\mathcal{G} = \mathcal{H}^t 0^r 3^{a-r} \underline{(1^r 0^{a-r} 2^r 1^{a-r} 0^r 2^{a-r})^t 1^r 0^r 3^{a-2r} 2^r},$$

   $\ell = c + a - r$ and $p = c + a$.
4. If $k \le r < a$, then

$$\mathcal{G} = \mathcal{H}^t 0^r 3^{a-r} \underline{1^r 0^{a-r} 2^r 1^{a-r} 0^r 2^{a-r}},$$

   $\ell = c + a - r$ and $p = 3a$ or $3a/2$.
5. If $r > 2a$, then $\mathcal{G} = \underline{\mathcal{H}^{t+1} 3^{r-2a}}$, $\ell = 0$ and $p = c + a$.

Remark 5. The pre-period and period of SUB(S) are not easy to determine, even if $S = \{s_1, s_2, s_3\}$ is a 3-element set. In [2, §4, Conjecture (i)], Althofer and Bultermann conjectured that the period of SUB(S) is bounded by a quadratic polynomial in s_3. Ho also studied SUB(S) for 3-element set S in [5].

4. The case S contains successive numbers

Proposition 4.1. *Let $S = \{a, a + 1, \dots, b - 1, b, c\}$ with $a < b < c$. Write $c = t(a+b)+r$ with $0 \le r < a+b$. Then*

$$\ell = 0, \quad p = \begin{cases} a + b, & a \le r \le b; \\ c + a, & r = 0 \textit{ or } r > b; \\ c + b, & 0 < r < a. \end{cases}$$

PROOF. Write $b = ak + s$, $0 \le s \le a - 1$ and denote by $\mathcal{H} = 0^a 1^a \cdots k^a (k + 1)^s$, then $\mathcal{G}_{\{a,a+1,\dots,b\}} = \underline{\mathcal{H}}$ with period $q = a + b = a(k + 1) + s$.

1. If $a \le r \le b$, then $\mathcal{G} = \underline{\mathcal{H}}$, $\ell = 0$ and $p = a + b$ by Lemma 2.1.
2. If $r = 0$, then

$$\mathcal{G} = \underline{\mathcal{H}^t (k + 1)^{a-s} (k + 2)^s}.$$

   If $r > b$ and $r + s > q$, then

$$\mathcal{G} = \underline{\mathcal{H}^{t+1} (k + 1)^{a-s} (k + 2)^{r+s-q}}.$$

   If $r > b$ and $r + s \le q$, then

$$\mathcal{G} = \underline{\mathcal{H}^{t+1} (k + 1)^{a+r-q}}.$$

   In all cases, we have $\ell = 0$ and $p = c + a$.

3. If $0 < r < a - 2s$, then

$$\mathcal{G} = \underline{\mathcal{H}^t, 0^r(k+1)^{a-s-r}(k+2)^s, 1^r(k+2)^{a-s-r}(k+3)^s, \cdots,} \\ \underline{(k-1)^r(2k)^{a-s-r}(2k+1)^s, k^r(2k+1)^s}.$$

If $a - 2s \le r < a - s$, then

$$\mathcal{G} = \underline{\mathcal{H}^t, 0^r(k+1)^{a-s-r}(k+2)^s, 1^r(k+2)^{a-s-r}(k+3)^s, \cdots,} \\ \underline{(k-1)^r(2k)^{a-s-r}(2k+1)^s, k^r(2k+1)^{a-s-r}(2k+2)^{2s+r-a}}.$$

If $a - s \le r < a$, then

$$\mathcal{G} = \underline{\mathcal{H}^t, 0^r(k+2)^{a-r}, 1^r(k+3)^{a-r}, \cdots (k-1)^r(2k+1)^{a-r}, k^r(k+1)^s},$$

In all cases, we have $\ell = 0$ and $p = c + b$.

5. Piecewise linearity of pre-periods and periods

Let S be a fixed subtraction set. Denote by $\mathcal{G}_S$ the nim-sequence of S with pre-period ℓ and period p. Denote by $\mathcal{G}_{S\cup\{c\}}$ the nim-sequence of $S \cup \{c\}$ with pre-period ℓ_c and period p_c. The following examples are due to computer-assistant calculations.

Example 5.1. *Let $S = \{6, 17\}$. Then $\mathcal{G}_S = \underline{0^6 1^6 0^5 2 1^5}$ with period 23. For $116 \le c \le 500$, we have*

$$\ell_c = \begin{cases} (9 - 2\lambda)c + (147 - 35\lambda), & c \equiv \lambda \text{ or } \lambda + 12 \bmod 23, \lambda \in [0, 4]; \\ 0, & \textit{otherwise}, \end{cases}$$

$$p_c = \begin{cases} c + 6, & c \equiv 0, 1, 2, 3, 4, 5, 12, 13, 14, 15, 16 \bmod 23; \\ c + 17, & c \equiv 7, 8, 9, 10, 11, 18, 19, 20, 21, 22 \bmod 23; \\ 23, & r = 6 \text{ or } 17. \end{cases}$$

See https://ruhuasiyu.github.io/nim/example5.1.html.

Example 5.2. *Let $S = \{3, 5, 8\}$. Then $\mathcal{G}_S = \underline{0^3 1^3 2^3 3^2}$ with period 11. For $13 \le c \le 500$, we have*

$$\ell_c = \begin{cases} c + 18, & c \equiv 1, 2 \bmod 11; \\ 0, & \textit{otherwise}, \end{cases} \qquad p_c = \begin{cases} c + 3, & c \equiv 0, 1, 9, 10 \bmod 11; \\ c + 25, & c \equiv 2 \bmod 11; \\ 11, & \textit{otherwise}. \end{cases}$$

See https://ruhuasiyu.github.io/nim/example5.2.html.

Example 5.3. *Let $S = \{2, 3, 5, 7\}$. Then $\mathcal{G}_S = \underline{0^2 1^2 2^2 3^2 4}$ with period 9. For $11 \le c \le 500$, we have*

$$\ell_c = \begin{cases} 2c - 4, & c \equiv 1 \bmod 18; \\ c + 5, & c \equiv 10 \bmod 18; \\ 0, & \textit{otherwise}, \end{cases} \qquad p_c = \begin{cases} c + 2, & c \equiv 0, 8, 9, 10, 17 \bmod 18; \\ 4, & c \equiv 1 \bmod 18; \\ 9, & \textit{otherwise}. \end{cases}$$

See https://ruhuasiyu.github.io/nim/example5.3.html.

Example 5.4. *Let* $S = \{4, 11, 12, 14\}$. *Then* $\mathcal{G}_S = \cdots \underline{20^4 1^4 0^3 3 1^3 2^3 0 3^3 12}$ *with pre-period* 24 *and period* 25. *Write* $r \equiv c \bmod 25, 0 \le r < 25$. *For* $101 \le c \le 500$*, we have*

$$\ell_c = \begin{cases} 4c+91, & r=0; \quad 2c+8, \quad r=1; \quad 2c+34, \quad r=2; \\ c-6, & r=3; \quad 2c+16, \quad r=4; \quad 2c+36, \quad r=5; \\ 3c+4, & r=6; \quad c+26, \quad r=9; \quad c+12, \quad r=12; \\ 0, & r=13; \quad 2c+37, \quad r=18; \quad c+14, \quad r=19; \\ c+2, & r=20; \quad 12, \quad r=21; \quad 3c+5, \quad r=22; \\ c+52, & r=23; \quad 2c+33, \quad r=24; \quad 24, \quad \text{otherwise}, \end{cases}$$

$$p_c = \begin{cases} c+37, & r=0,1,9,18; \quad c+14, \quad r=2,10; \\ c+11, & r=6,7,8,15,16,17; \quad c+12, \quad r=13; \\ 2c+41, & r=19; \quad c+4, \quad r=21; \\ c+28, & r=22; \quad 25, \quad \text{otherwise}. \end{cases}$$

See https://ruhuasiyu.github.io/nim/example5.4.html.

Based on these observations, we propose the Conjecture 1.3. By the results in §§2-4, Conjecture 1.3 is valid in the cases mentioned in Theorem 1.4.

Proof (Proof of Theorem 1.4).

1. The period of $\text{SUB}(S)$ is $q = 2$. If c is odd, then $\mathcal{G}_{S\cup\{c\}} = \underline{01}$. If c is even, denote by s the maximal number in S. Then

$$\mathcal{G}_{S\cup\{c\}} = \underline{(01)^{c/2}(23)^{(s-1)/2}2},$$

$\ell_c = 0$ and $p_c = c + s$.
2. follows from Example 2.4 and Proposition 2.5.
3. follows from Proposition 3.1.
4. follows from Proposition 4.1.

6. Ultimately bipartite nim-sequences

A subtraction game (or its nim-sequence) is said to be *ultimately bipartite* if the period is 2. It is known that $\mathcal{G}_S$ is ultimately bipartite with pre-period 0 if and only if $1 \in S$ and all elements in S are odd, see [4].

Example 6.1. *Let* $a \ge 3$ *be an odd integer. If* S *is one of the following:*

- $S = \{3, 5, 9, \dots, 2^a + 1\}$;
- $S = \{3, 5, 2^a + 1\}$;
- $S = \{a, a + 2, 2a + 3\}$;
- $S = \{a, 2a + 1, 3a\}$,

then $\text{SUB}(S)$ *is ultimately bipartite. See [4, Theorem 2] and [5, Theorem 5].*

Lemma 6.2. *If* $\mathcal{G} = \mathcal{G}_S$ *is ultimately bipartite, then all elements in* S *are odd.*

PROOF. As shown in [4, Theorem 3], there exists an integer n_0 such that for $n \geq n_0$, $\mathcal{G}(n) = 0$ if n is even; $\mathcal{G}(n) = 1$ if n is odd. Take an even number $n \geq n_0 + s_k$, where s_k is the maximal element in S. Then

$$0 = \mathcal{G}(n) = \text{mex}\,\{\mathcal{G}(n-s) : s \in S\},$$

which implies that $\mathcal{G}(n-s) = 1$ for all $s \in S$. Hence all $s \in S$ are odd.

We have the following new ultimately bipartite subtraction sets inspired by our conjecture.

Theorem 6.3. *Let $a \geq 3$ be an odd integer and $t \geq 1$. The subtraction game* SUB(S) *is ultimately bipartite in the following cases:*

1. $S = \{a, a+2, (2a+2)t+1\}$;
2. $S = \{a, 2a+1, (3a+1)t-1\}$;
3. $S = \{a, 2a-1, (3a-1)t+a-2\}$.

PROOF. Let c be the maximal element in S. Write $a = 2k+1$.

1. If $k \geq 2$, then the leading $(k+1)(a+1)(2t+1)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((a+1)(2t+1)i+j\big), 0 \leq j < (a+1)(2t+1) = c+a$			
0	0^a1	[$1^{a-1}22$ $1^{a-1}22$	0^a1 $02^{a-3}331$	$]^{t-1}$,
1	$030^{a-2}1$	[$01^{a-2}21$ $01^{a-2}21$	$020^{a-2}1$ $0202^{a-5}321$	$]^{t-1}$,
i	$(01)^{i-1}030^{a-2i}1$	[$(01)^{i-1}01^{a-2i}21$ $(01)^{i-1}01^{a-2i}21$	$(01)^{i-1}020^{a-2i}1$ $(01)^{i-1}0202^{a-2i-3}321$	$]^{t-1}$,
$k-1$	$(01)^{k-2}030^31$	[$(01)^{k-2}01^321$ $(01)^{k-2}01^321$	$(01)^{k-2}020^31$ $(01)^{k-2}020321$	$]^{t-1}$,
k	[$(01)^{k-1}0301$	$(01)^{k-1}0121]^{t-1}$ $(01)^{k-1}0101$	$(01)^{k-1}03\underset{\sim}{01}$, $(01)^{k-1}0101$.	

Hence the pre-period is

$$\ell = (k+1)(c+a) - 2a - 4 = (k+1)c + 2k^2 - k - 5$$

and the period is $p = 2$. The case $a = 3$ will be shown in Case 3.

2. The leading $(k+1)((3a+1)t+a-1)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big(((3a+1)t+a-1)i+j\big), 0 \leq j < (3a+1)t+a-1 = c+a$				
0	[0^a	1^a	$02^{a-1}1$	$]^t$,	3^{a-1}
1	[020^{a-2} 020^{a-2}	101^{a-2} 101^{a-2}	$(01)^132^{a-3}1$ $(01)02^{a-3}1$	$]^{t-1}$,	 $(01)3^{a-3}$
i	[$(01)^{i-1}020^{a-2i}$ $(01)^{i-1}020^{a-2i}$	$1(01)^{i-1}01^{a-2i}$ $1(01)^{i-1}01^{a-2i}$	$(01)^i32^{a-2i-1}1$ $(01)^i02^{a-2i-1}1$	$]^{t-1}$,	 $(01)^i3^{a-2i-1}$
$k-1$	[$(01)^{k-2}020^3$ $(01)^{k-2}020^3$	$1(01)^{k-2}01^3$ $1(01)^{k-2}01^3$	$(01)^{k-1}32^21$ $(01)^{k-1}02^21$	$]^{t-1}$,	 $(01)^{k-1}3^2$
k	[$(01)^{k-1}020$ $(01)^{k-1}02\underset{\sim}{0}$	$1(01)^{k-1}01$ $\underset{\sim}{1}(01)^{k-1}01$	$(01)^k31$ $(01)^k01$	$]^{t-1}$,	 $(01)^k$.

Hence the pre-period is

$$\ell = (k+1)(c+a) - 3a - 1 = (k+1)c + 2k^2 - 3k - 3$$

and the period is $p = 2$.

(3) The leading $(k+1)(3a-1)(t+1)$ terms of $\mathcal{G}$ are

i	$\mathcal{G}\big((3a-1)(t+1)i+j\big), 0 \le j < (3a-1)(t+1) = c+2a+1$					
0	$[0^{a-1}$	01^{a-1}	$12^{a-1}\ ]^t,$	$0^{a-2}3$	$31^{a-3}(10)^1$	$2^{a-2}(01)^1$
1	$[0^{a-3}(01)^1$	$31^{a-3}(10)^1$	$2^{a-2}(01)^1\,]^t,$	$0^{a-4}3(01)^1$	$31^{a-5}(10)^2$	$2^{a-4}(01)^2$
i	$[0^{a-2i-1}(01)^i$	$31^{a-2i-1}(10)^i$	$2^{a-2i}(01)^i]^t,$	$0^{a-2i-2}3(01)^i$	$31^{a-2i-3}(10)^{i+1}$	$2^{a-2i-2}(01)^{i+1}$
$k-1$	$[0^2(01)^{k-1}$	$31^2(10)^{k-1}$	$2^3(01)^{k-1}\,]^t,$	$0^1 3(01)^{k-1}$	$3(10)^k$	$2^1(01)^k$
k	$[(01)^k$	$3(10)^k$	$2(\underset{\sim}{01})^k\ \ ]^{t-1},(01)^{6k+2}.$			

Hence the pre-period is

$$\ell = (k+1)(c+2a+1) - 2(7k+2) = (k+1)c + 4k^2 - 7k - 1$$

and the period is $p = 2$.

Remark 6. One may expect that if $\text{SUB}(a,b,c)$ is ultimately bipartite, then so is $\text{SUB}(a,b,d)$ for sufficient large d with $d \equiv c \mod (a+b)$. This is not true in general. For example, $\text{SUB}(3,11,13)$ is ultimately bipartite but $\text{SUB}(3,11,14t+13)$ has period $14t+16$, $t \ge 1$.

Remark 7. Write $a = 2k+1$. Consider the four-element subtraction set $S = \{a, 2a+1, 3a, c\}$ with odd $c > 3a$. For $3 \le a \le 25, c < 500$, we find the following phenomenon.

- If $c = 4a+1$, then $\ell = 0$ and $p = 5a+1$.
- If $c = (4i+2)a - 1$ with $1 \le i < k$, then $\ell = (8i-1)a + 2i - 1$ and $p = 4a$.
- Otherwise, $\text{SUB}(S)$ is ultimately bipartite.

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